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Interface Motion in a Planar Spin-Flip Model Derived from Exclusion on the Line

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We consider a two-dimensional spin-flip model, which can be interpreted as the limit of the Ising model at low temperature and a small nonzero external field. In the hydrodynamic limit and for a special class of initial conditions, the motion of the interface is governed by a nonlinear partial differential equation with a lattice-distorted curvature term. In our proofs we use results about the hydrodynamic behavior of the weakly asymmetric exclusion process on the integers and also on the nonnegative integers with a trap at the boundary.

KEY WORDS: Glauber dynamics; hydrodynamic limit; exclusion process; motion by curvature.

1. INTRODUCTION

For a spin system with a given Hamiltonian one often introduces rates for flipping the spins one at a time in such a way that every Gibbs state becomes reversible. The application of this Glauber or spin-flip dynamics to a nonequilibrium state may serve as a model for the behavior of the system under the physical conditions described by the Hamiltonian.

In this paper we will study a class of attractive nearest-neighbor spin-flip models with state space $\{+1, -1\}^{\mathbb{Z}^2}$ for which flips occur only at sites having at least two nearest neighbors with opposite spin. Rates satisfying these assumptions appear, for example, as the low-temperature limit of rates for the ferromagnetic Ising model with nearest-neighbor interaction and with a constant external field that is decreased linearly with the absolute temperature.

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We will concentrate on the asymmetric case when the rate for a $(+)$ to $(-)$ flip at sites with exactly two positive neighbors is different from the $(-)$ to $(+)$ rate. This corresponds to a nonzero external field in the Ising models. Our aim is to find the equation governing the motion of the interface separating the plus from the minus region on a macroscopic scale. To this end, we consider the spin-flip process on the lattice $(\varepsilon\mathbb{Z})^2$, send the asymmetry of the rates to zero linearly with $\varepsilon > 0$, and choose an appropriate time scale. In this continuum limit, let the initial interface be given by some deterministic smooth curve $\{x, \varphi_0(x)): x \in \mathbb{R}\}$ with positive spins above the curve (i.e., in $\{x, y): y > \varphi_0(x)\}$) and negative spins below it. We want to prove that time evolution deforms this curve into $\{(x, \varphi(t, x)): x \in \mathbb{R}\}$ at macroscopic time t, where $\varphi(t, \cdot)$ is the solution of the following initial-boundary-value problem:

$$\varphi_t = \tfrac{1}{2}\varphi_{xx}(1 + |\varphi_x|)^{-2} + \gamma\, |\varphi_x|\, (1 + |\varphi_x|^{-1}, \qquad t > 0, \quad x \in \mathbb{R} \qquad (1.1)$$
$$\varphi(t, x) \to \varphi_0(x) \qquad \text{as} \quad t \to 0^+ \quad \text{for all} \quad x \in \mathbb{R}$$

Here subscripts denote partial derivatives in the (t, x) plane. The parameter $\gamma \neq 0$ expresses the asymmetry of the rates. Its sign is opposite to that of the external field in the Ising models.

The initial condition is written as a limit statement because later we allow discontinuous initial functions, possibly with infinite values.

We prove Eq. (1.1) rigorously in two special cases.

1. Nonincreasing φ_0, which are constant for sufficiently large x. We allow what we will call "pinned stairstep configurations" to approximate φ_0 with their interfaces.

2. Unimodal φ_0 with $\varphi_0(x) = -x$ for $x \leqslant 0$, $\varphi_0(x)$ nondecreasing for nonnegative x. Here we restrict the initial configurations to so-called "V-stairsteps" with a unimodal interface and freeze the dynamics on its decreasing part. The limit interface $\varphi(t, \cdot)$ will be unimodal for all $t > 0$. On its increasing part, (1.1) is satisfied.

The basic idea of proof is a mapping to one-dimensional exclusion models. These are particle system on $\mathbb{Z}$ where particles change places with neighboring holes.

Equations (1.1) follows from hydrodynamic limit theorems for the exclusion process. In the case of pinned monotone interfaces we can directly use known results[2,4,6] where Burgers' equation with viscosity is obtained. Partly frozen interfaces lead to an exclusion process with boundary condition treated in the thesis of the author[12] using a nonlinear transformation of the particle system in the spirit of Gärtner.[4]

Standard references for the derivation of hydrodynamic equations are refs. 3 and 10.

It is tempting to compare Eq. (1.1) to known results for the symmetric case (vanishing external field and thus $\gamma = 0$). Spohn[11] has shown that the equation

$$\varphi_t = \tfrac{1}{2}\varphi_{xx}(1 + |\varphi_x|)^{-2}, \qquad t > 0, \quad x \in (-1, 1]$$

governs the macroscopic behavior of a one-dimensional zero-range process with two components of particles annihilating each other and periodic boundary conditions. This process describes the evolution of the interface height $\varphi(t, x)$ above the x axis as a function of x and time t when one applies spin-flip dynamics of the Ising model at zero temperature and vanishing external field while forbidding flips which would render the interface height function multivalued.

Spohn also gives a geometric interpretation; the embedded curve in $\mathbb{R}^2$ has normal velocity

$$v_n = (\kappa/2)(|\sin \vartheta| + |\cos \vartheta|)^{-2}$$

where κ is the local curvature and ϑ the angle between the tangent vector and the x axis (i.e., $\tan \vartheta = \varphi_x$); cf. Eq. (4.26) in ref. 11.

The same lattice-distorted curvature term is also found in Eq. (1.1). The additional first-order term accounts for the asymmetry of the rates. It depends only on the limit asymmetry γ and the angle ϑ through $|\varphi_x|$.

The main results of this paper are Theorems 2.1 and 2.2. There, the parametrization of the interface is adapted to our mapping to the exclusion model. In Sections 2.1 and 2.2 we explain this mapping and show how to obtain (1.1) from our theorems. We summarize the necessary facts about the hydrodynamic behavior of the exclusion model in Section 3. The proofs in Section 4 heavily rely on them.

2. THE SPIN-FLIP MODEL AND ITS MAPPING TO THE EXCLUSION PROCESS

The spin-flip models we are going to investigate are Markov processes $(\omega_t)_{t \geq 0}$ with state spate $\{-1, +1\}^{\mathbb{Z}^2}$.

In all our models, the spin $\omega(p)$ at any site $p \in \mathbb{Z}^2$ flips to $-\omega(p)$ at a rate depending only on $\omega(p)$, the number $s(p, \omega)$ of neighbor sites q of p with $\omega(q) = -\omega(p)$, and a parameter $\alpha \neq 0$. In infinitesimal time intervals, flips at different sites occur independently.

Denoting the flipping rate at p by $c^{\alpha}(\omega(p), s(p, \omega))$, we can express this heuristic description of the process in terms of its generator. Applied to a cylinder function f, it has the value

$$\sum_{p \in \mathbb{Z}^2} c^{\alpha}(\omega(p), s(p, \omega))(f(\omega^{(p)} - f(\omega))$$

at $\omega \in \{-1, +1\}^{\mathbb{Z}^2}$, where $\omega^{(p)}$ denotes the configuration obtained from ω by flipping the spin at p.

The numbers $c^{\alpha}(\sigma, s)$, $\sigma \in \{-1, +1\}$, $s \in \{0, 1, 2, 3, 4\}$, are supposed to satisfy

$$\begin{aligned} 0 = c^{\alpha}(\sigma, 0) = c^{\alpha}(\sigma, 1) < c^{\alpha}(\sigma, 2) &= \frac{1 + \sigma \cdot \tanh(\alpha)}{2} \\ &\leqslant c^{\alpha}(\sigma, 3) \leqslant c^{\alpha}(\sigma, 4) \leqslant +\infty \end{aligned} \tag{2.1}$$

This condition is satisfied for low-temperature limits of certain spin-flip dynamics associated to the two-dimensional Ising model in the following sense: Consider the ferromagnetic nearest-neighbor Ising model in the square lattice with a nonvanishing constant external field $h \in \mathbb{R} \backslash \{0\}$, inverse temperature $\beta > 0$, and interaction strength $J > 0$ between nearest neighbors. Its formal Hamiltonian reads

$$H(\omega) = -\beta J \sum_{\langle p, q \rangle} \omega(p)\, \omega(q) - \beta h \sum_{p \in \mathbb{Z}^2} \omega(p), \qquad \omega \in \{-1, +1\}^{\mathbb{Z}^2}$$

where the first sum is over (nearest) neighbor pairs of sites in $\mathbb{Z}^2$.

Let us now suppose that strictly positive numbers $c(\sigma, s)$, $\sigma \in \{-1, +1\}$, $s \in \{0, 1, 2, 3, 4\}$, have the property

$$c(+1, s) = c(-1, s)\, e^{-2\beta J(4-2s)} e^{-2\beta h}, \qquad s = 0, 1, 2, 3, 4$$

and take $c(\omega(p), s(p, \omega))$ as the rate for flipping the spin at p in a configuration ω.

These rates satisfy the detailed balance condition with respect to H. Hence the corresponding spin-flip process is reversible if it is started in any H-Gibbs state. One such example was discussed by Marchand and Martin,[8] others can be derived specializing the rates mentioned in Liggett,[7] Chapter IV.2. In the low-temperature limit

$$\beta \to +\infty, \quad h \to 0, \quad \alpha = -\beta h \neq 0 \text{ fixed}, \quad J > 0 \text{ fixed}$$

(i.e., the external field is made proportional to the temperature $1/\beta$), these three choices all satisfy (2.1) up to a constant factor.

The fact that each spin of the sign of H gives the negative energy contribution $-\beta\,|h|$ to the Hamiltonian is reflected in the dynamics as

$$c(\operatorname{sgn}(h), 2) < c(-\operatorname{sgn}(h), 2)$$

i.e., a spin of $\operatorname{sgn}(h)$ is less likely to disappear in a short time than a spin of $-\operatorname{sgn}(h)$ if both have two neighbor spins of either sign.

Define the *interface* of a configuration $\omega \in \{+1, -1\}^{\mathbb{Z}^2}$ as the boundary (in $\mathbb{R}^2$) of the union of all unit cubes of the form $[p_1 - 1, p_1] \times [p_2 - 1, p_2]$ taken over all $(p_1, p_2) \in \mathbb{Z}^2$ with $\omega(p_1, p_2) = +1$. It is always contained in

$$\{(x, y) \in \mathbb{R}^2 \colon x \text{ or } y \text{ is an integer}\}$$

In order to obtain (1.1) in the continuum limit, set for positive ε

$$\alpha = \alpha(\varepsilon) := \gamma\varepsilon \qquad \text{for some fixed} \quad \gamma \neq 0$$

fix a family $(c^{\alpha(\varepsilon)}(\cdot, \cdot))_{\varepsilon > 0}$ of rates satisfying (2.1), and call the corresponding family of spin-flip processes $((\omega_t^\varepsilon)_{t \geqslant 0})_{\varepsilon > 0}$. Rescale their interfaces to

$$\{x, y) \in \mathbb{R}^2 \colon x/\varepsilon \text{ or } y/\varepsilon \text{ is an integer}\}$$

Our aim is to prove statements of the following type:

If the rescaled interfaces of the initial configurations ω_0^ε have a simple structure and converge (as $\varepsilon \to 0$) to the curve $\{(x, \varphi_0(x))\}$, then the rescaled interfaces of $\omega_{t/\varepsilon^2}^\varepsilon$ converge to $\{x, \varphi(t, x))\}$, where $\varphi(t, x)$ is the solution of the Cauchy problem (1.1).

Below we are going to state Theorem 2.1 for what we shall call "pinned stairstep configurations" and Theorem 2.2 for special configurations with nonmonotone interface under a partially frozen dynamics. We also explain the maps to the corresponding version of the exclusion process in these subsections.

2.1. Pinned Stairstep Configurations

Call $\omega \in \{-1, +1\}^{\mathbb{Z}^2}$ a *stairstep configuration* if its interface is nonempty and the spins right and above any lattice site with positive spin are also positive, i.e., for all $p_1, p_2 \in \mathbb{Z}$ and $l \in \mathbb{N} = \{1, 2, ...\}$ it holds that

$$\omega(p_1, p_2 + l) \geqslant \omega(p_1, p_2) \qquad \text{and} \qquad \omega(p_1 + l, p_2) \geqslant \omega(p_1, p_2)$$

It is easy to see that $s(p, \omega) \in \{0, 1, 2\}$ for all sites p in any stairstep configuration ω. Since our conditions (2.1) on c^α forbid flips with $s = 0$ or

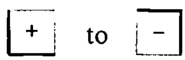
$s=1$, the set of stairstep configurations is invariant under c^α-dynamics and, moreover, only events with $s=2$ occur.

Looking again at (2.1), this shows that the evolution of stairstep configurations is completely determined by α and does not depend on the particular choice of $c^\alpha(\sigma, s)$ for $s=3, 4$.

In Fig. 1 we give a typical example of a stairstep configuration. Our convention is that the sign in a plaquette stands for the spin at its upper right corner.

The interface of a stairstep configuration is always simply connected and, if it is not empty, its shape resembles an infinitely long stairstep.

Under c^α-dynamics, only spins at the corners of the interface can flip. The rate for flipping



is $c^\alpha(+1, 2) = [1+\tanh(\alpha)]/2$, for the reverse direction it is $c^\alpha(-1, 2) = [1-\tanh(\alpha)]/2$.

Our theorem on the macroscopic evolution of interfaces will be formulated for initial distributions living on a subset PS of $\{-1, +1\}^{\mathbb{Z}^2}$ which we shall call *pinned stairstep configurations*. By definition, a stairstep configuration is pinned iff there is an integer N such that for all $p_1 \geqslant N$

$$\omega(p_1, p_2) = \begin{cases} +1 & \text{if } p_2 > 0 \\ -1 & \text{if } p_2 \leqslant 0 \end{cases}$$

Loosely speaking, this means that its interface coincides with the x axis for x large enough. The advantage of pinned interfaces is that their position in $\mathbb{Z}^2$ is determined by the increments of the interface height. This will be important because our mapping to the one-dimensional model below will be one-to-one restricted to pinned stairstep configurations.

Our example above is pinned if we assume that the interface height stays zero for x right of the area shown in the figure.

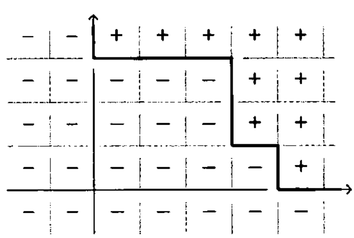


Fig. 1. A pinned stairstep configuration.

Considering the family $((\omega_t^\varepsilon)_{t\geq 0})_{\varepsilon>0}$ of processes as introduced above, it is clear that $\omega_t^\varepsilon \in PS$ for all $t\geq 0$ if ω_0^ε is distributed on PS. In this case and for arbitrary $x\in \varepsilon\mathbb{Z}$ and $t\geq 0$,

$$\varphi^\varepsilon(t, x) := \sup\{ y\in \varepsilon\mathbb{Z}: \omega_{t/\varepsilon^2}^\varepsilon(x/\varepsilon, y/\varepsilon) = -1\}$$

(with the convention $\sup \varnothing = +\infty$) is a well-defined quantity from $(-\infty, +\infty]$ describing the macroscopic height of the interface above the x-axis in the space-time rescaling under consideration.

Since $\omega_0^\varepsilon \in PS$, $\varphi^\varepsilon(t, x)$ is nonincreasing in x. Hence any limit curve will be nonincreasing. Instead of (1.1) we can therefore write

$$\begin{aligned} &\varphi_t = \tfrac{1}{2}\varphi_{xx}(1-\varphi_x)^{-2} - \gamma\varphi_x(1-\varphi_x)^{-1}, \qquad t>0, \quad x\in\mathbb{R} \\ &\varphi(t,x)\to\varphi_0(x) \qquad \text{as} \quad t\to 0^+ \quad \text{for all} \quad x\in\mathbb{R} \end{aligned} \tag{2.2}$$

As announced in the introduction, we are going to formulate our result in a different coordinate system. We will use the lattice length a (and time t) to parametrize the interface. This is not only "more geometric," but also crucial for our proof of the equation in its simpler form in (t, a) coordinates.

To this end, order the lattice points on the interface of an arbitrary pinned stairstep ω_t^ε into a doubly infinite sequence such that subsequent points are nearest neighbors on $\mathbb{Z}^2$. There is exactly one way to index this sequence as

$$(p_1(l, \omega), p_2(l, \omega))_{l\in\mathbb{Z}}$$

such that it satisfies

$$l = p_1(l, \omega) - p_2(l, \omega) \qquad \text{for all} \quad l\in\mathbb{Z} \tag{2.3}$$

For the process $(\omega_t^\varepsilon)_{t\geq 0}$ we rescale its interface to $(\varepsilon\mathbb{Z})^2$, i.e., we consider for $a\in\mathbb{Z}$ and $t\geq 0$ the macroscopic coordinates of the interface

$$(x^\varepsilon(t, a), y^\varepsilon(t, a)) := (\varepsilon\cdot p_1(a/\varepsilon, \omega_{t/\varepsilon^2}^\varepsilon), \varepsilon\cdot p_2(a/\varepsilon, \omega_{t/\varepsilon^2}^\varepsilon))$$

Then we have

$$a = x^\varepsilon(t, a) - y^\varepsilon(t, a) \qquad \text{for all} \quad a\in\varepsilon\mathbb{Z} \tag{2.4}$$

It is obvious that a is the lattice length parameter: The segment of the interface connecting the points indexed by a_1 and a_2 has length $|a_1 - a_2|$ for any $a_1, a_2\in\varepsilon\mathbb{Z}$.

The relation between $\varphi^\varepsilon(t, x)$ and this sequence is given by

$$\varphi^\varepsilon(t, x) = \sup\{ y^\varepsilon(t, a): a\in\varepsilon\mathbb{Z}, x^\varepsilon(t, a) = x\}, \qquad t\geq 0, \quad x\in\varepsilon\mathbb{Z}$$

In the nonorthogonal (a, y) coordinates, $y^\varepsilon(t, a)$ can also be regarded as a rescaled height. Further on, we will work mainly with $y^\varepsilon(t, a)$. The full point sequence can be recovered using (2.4).

If we do the analogous thing on $\mathbb{R}^2$, i.e., introduce

$$a := x - y$$

and reparametrize the interface $(x, \varphi(t, x))_{x \in \mathbb{R}}$ as

$$(x(t, a), y(t, a))_{a \in \mathbb{R}} \qquad \text{with} \quad x(t, a) - y(t, a) = a$$

for every time $t \geqslant 0$, then (2.2) becomes

$$\frac{\partial y}{\partial t}(t, a) = \frac{1}{2}\frac{\partial^2 y}{\partial a^2}(t, a) - \gamma \cdot \frac{\partial y}{\partial a}(t, a) - \gamma\left(\frac{\partial y}{\partial a}(t, a)\right)^2, \qquad t > 0, \quad a \in \mathbb{R} \quad \text{(2.5a)}$$

$$y(t, a) \to y_0(a) \qquad \text{for every} \quad a \in \mathbb{R} \quad \text{as} \quad t \to 0^+ \qquad \text{(2.5b)}$$

To obtain this, express the total differential $dy = d\varphi$ in both the (t, x) and (t, a) coordinate systems, check

$$\frac{\partial \varphi}{\partial x}(t, x) = \left(\frac{\partial x}{\partial a}(t, a)\right)^{-1}\left(1 - \frac{\partial x}{\partial a}(t, a)\right)$$

using $a = x - y$, and compute the dx part of the total differential of $(\partial\varphi/\partial x)(t, x)$. With the obtained relations one easily passes from (2.2) to (2.5).

Note that a is *not* the length parameter on the interface curve at time t.

Formulating our theorem in the $y(t, a)$ picture permits a slightly more general formulation because φ_0 may have infinite valued and jumps even when y_0 is finite-valued and continuous. See the example following the theorem.

Note that Eq. (2.5) can be transformed to the heat equation: Set

$$\begin{aligned} w(t, a) &:= \exp[-2\gamma y(t, a) - \gamma a + \gamma^2 t/2], \qquad t > 0, \quad a \in \mathbb{R} \\ w_0(a) &:= \exp[-2\gamma y_0(a) - \gamma a], \quad a \in \mathbb{R} \end{aligned} \qquad \text{(2.6)}$$

Then it becomes

$$\begin{aligned} &\frac{\partial w}{\partial t}(t, a) = \frac{1}{2}\frac{\partial^2 w}{\partial a^2}(t, a), \qquad t > 0, \quad a \in \mathbb{R} \\ &w(t, a) \to w_0(a) \qquad \text{for every} \quad a \in \mathbb{R} \quad \text{as} \quad t \to 0 \end{aligned} \qquad \text{(2.7)}$$

As a by-product we obtain existence and uniqueness of a classical solution to (2.5) for every continuous y_0: $\mathbb{R} \to \mathbb{R}$ satisfying

$$\sup_{a \in \mathbb{R}} |y_0(a)|/(1+|a|) < +\infty$$

in the class of solutions with

$$\sup_{0 \leqslant t \leqslant T} \sup_{a \in \mathbb{R}} y(t, a)|/(1+|a|) < +\infty \qquad \text{for every} \quad T > 0$$

Explicitly, this solution reads

$$y(t, a) = -\frac{a}{2} - \frac{\gamma t}{4} - \frac{1}{2\gamma} \cdot \log \left\{ \frac{1}{(2\pi t)^{1/2}} \int_{-\infty}^{\infty} \exp\left[-\frac{(z-a)^2}{2t} - 2\gamma y_0(z) - \gamma z \right] dz \right\} \tag{2.8}$$

for $t > 0$ and $a \in \mathbb{R}$.

Theorem 2.1. Let ω_0^ε, $\varepsilon > 0$, be random initial configurations from *PS* with the property that their rescaled height $y^\varepsilon(0, \cdot)$ converge vaguely in probability to a given piecewise differentiable function y_0: $\mathbb{R} \to \mathbb{R}$, i.e.,

$$\varepsilon \sum_{a \in \varepsilon \mathbb{Z}} f(a)\, y^\varepsilon(0, a) \xrightarrow{\mathbb{P}} \int_{-\infty}^{\infty} f(a)\, y_0(a)\, da \qquad \text{for} \quad \varepsilon \to 0^+ \tag{2.9}$$

for all f from the space C_0 of continuous functions on $\mathbb{R}$ with compact support. Additionally, we assume that for each $\delta > 0$ there is an $\varepsilon_0 > 0$ such that

$$\sup_{\varepsilon \geqslant \varepsilon_0} \mathbb{P}(y^\varepsilon(0, a) > \delta) \to 0 \qquad \text{as} \quad a \to \infty \tag{2.10}$$

Then $y^\varepsilon(t, a)$ converges vaguely in probability to the solution $y(t, a)$ of (2.5) for all $t > 0$ and the convergence is uniform in $[0,T]$ for all finite $T > 0$. For arbitrary $f \in C_0$, $T > 0$, and $\delta > 0$ this means

$$\mathbb{P}\left(\sup_{0 \leqslant t \leqslant T} \left| \varepsilon \sum_{a \in \varepsilon \mathbb{Z}} f(a)\, y^\varepsilon(t, a) - \int_{-\infty}^{\infty} f(a)\, y(t, a)\, da \right| > \delta \right) \to 0 \qquad \text{as} \quad \varepsilon \to 0^+ \tag{2.11}$$

Remarks. 1. Piecewise differentiability of y_0 means that y_0 is continuous but the set of a where $y_0(a)$ is not differentiable has no (finite) accumulation points.

2. Our restriction to PS-configurations implies

$$y^\varepsilon(0, a+\varepsilon) - y^\varepsilon(0, a) \in \{0, -\varepsilon\} \qquad \text{for} \quad a \in \mathbb{Z}$$

and

$$y^\varepsilon(0, a) = 0 \qquad \text{for} \quad a \in \varepsilon\mathbb{Z} \quad \text{large enough}$$

so that $y^\varepsilon(0, a)$ is nonnegative and nonincreasing in $a \in \varepsilon\mathbb{Z}$. For every y_0 permitting an approximation by PS-interfaces as in (2.9) it follows that $y_0(a)$ is a nonnegative nonincreasing function of $a \in \mathbb{R}$ satisfying

$$\frac{dy_0}{da}(a) \in [-1, 0] \qquad \text{for almost all} \quad a \in \mathbb{R}$$

The "one-sided" tightness conditions (2.10) implies $y_0(a) \to 0$ as $a \to \infty$.

Example. Let $\bar{\omega}$ be the configuration with positive spins at all sites in $\mathbb{Z}^2$ with both coordinates positive, all other spins negative. Set ω_0^ε to $\bar{\omega}$ deterministically. The interface of $\bar{\omega}$ is then

$$\{(x, y) \in \mathbb{R}^2 \colon (x = 0 \text{ and } y \geqslant 0) \text{ or } (y = 0 \text{ and } x \geqslant 0)\}$$

Parametrizing it by $a = x - y$ gives

$$y^\varepsilon(0, a) = \begin{cases} -a & \text{for} \quad a \in \varepsilon\mathbb{Z}, \quad a < 0 \\ 0 & \text{for} \quad a \in \varepsilon\mathbb{Z}, \quad a \geqslant 0 \end{cases}$$

and

$$y_0(a) = \begin{cases} -a & \text{for} \quad a < 0 \\ 0 & \text{for} \quad a \geqslant 0 \end{cases}$$

Plugging this into formula (2.8) and going to the limit as $t \to \infty$, we obtain in the case $\gamma < 0$ that $y(t, a)$ tends to

$$y_\infty(a) := -\frac{1}{2\gamma} \cdot \log(1 + e^{2\gamma a}), \qquad a \in \mathbb{R}$$

Thus, the asymptotic interface is given by

$$\{(x, y) \colon e^{2\gamma x} + e^{2\gamma y} = 1, x \geqslant 0, y \geqslant 0\}$$

This result is in agreement with earlier work of Marchand and Martin,[8] who considered the c^α-process starting from $\bar{\omega}$.

In our notation (with α different from that in ref. 8) they proved for $\alpha < 0$ that the distribution after time t converges (for $t \to \infty$) to some probability measure $\mathbb{P}^\alpha$ on $\{+1, -1\}^{\mathbb{Z}^2}$ with $\mathbb{P}^\alpha(PS) = 1$. After rescaling the interface to the ε-lattice $(\varepsilon\mathbb{Z})^2$ and letting $\varepsilon \to 0$, Marchand and Martin showed that the distribution of this rescaled interface under $\mathbb{P}^{\alpha(\varepsilon)}$ converges to the measure putting unit mass on the graph above.

Their proof uses a mapping to the one-dimensional exclusion process, knowledge about the stationary measures of the latter, and combinatorics. It inspired us to extend the same method to pinned stairstep configurations and their time evolution.

Next we introduce the main tool for the proof of this theorem, the mapping to the one-dimensional exclusion process on $\mathbb{Z}$ with state space $\{0, 1\}^{\mathbb{Z}}$. Take any $\omega \in PS$ and construct its point sequence $(p_1(l, \omega), p_2(l, \omega))_{l \in \mathbb{Z}}$ as before. We define the image $\zeta = (\zeta(l))_{l \in \mathbb{Z}}$ of ω by

$$\zeta(l) := p_2(l-1, \omega) - p_2(l, \omega), \qquad l \in \mathbb{Z}$$

The properties of a pinned stairstep imply that ζ belongs to the set

$$FR := \left\{ \zeta = (\zeta(l))_{l \in \mathbb{Z}} \in \{0, 1\}^{\mathbb{Z}} : \sum_{l=1}^{\infty} \zeta(l) < +\infty \right\}$$

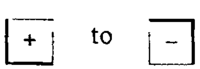
Interpreting $\zeta \in \{0, 1\}^{\mathbb{Z}}$ as a particle configuration [$\zeta(l) = 1$ if $l \in \mathbb{Z}$ is occupied, $\zeta(l) = 0$ if empty], we have that FR is the set of configurations with only finitely many particles on the right.

This map is a bijection between PS and FR: The only $\omega \in PS$ being mapped to some $\zeta \in FR$ is the configuration having the interface obtained by connecting the neighboring points in the sequence

$$\left(l + \sum_{k=l+1}^{\infty} \zeta(k), \sum_{k=l+1}^{\infty} \zeta(k) \right), \qquad l \in \mathbb{Z} \tag{2.12}$$

by line segments. The sums are finite due to $\zeta \in FR$. Note the analogy to (2.3).

Now apply this bijection to map the paths of the spin-flip process $(\omega_t)_{t \geqslant 0}$, started almost surely in PS, to a process $(\zeta_t)_{t \geqslant 0}$ living on the subset FR of $\{0, 1\}^{\mathbb{Z}}$. A flip of the type



is mapped to an exchange

$$\dots 10 \dots \quad \text{to} \quad \dots 01 \dots$$

in ζ written as a doubly infinite 0–1 sequence. Analogously, flips in the reverse direction are seen in the FR picture as transitions

$$\ldots 01 \ldots \quad \text{to} \quad \ldots 10 \ldots$$

Thus we obtain a process $(\zeta_t)_{t \geqslant 0}$ whose generator, applied to a cylinder function f on $\{0, 1\}^{\mathbb{Z}}$, has the value

$$\sum_{k \in \mathbb{Z}} \{c^\alpha(+1, 2) \cdot \zeta(k)[1 - \zeta(k+1)] \\ + c^\alpha(-1, 2) \cdot \zeta(k+1)[1 - \zeta(k)]\} [f(\zeta^{k, k+1}) - f(\zeta)]$$

at $\zeta \in FR \subseteq \{0, 1\}^{\mathbb{Z}}$, where $\zeta^{k, k+1}$ is the element of FR obtained from ζ by exchanging $\zeta(k)$ and $\zeta(k+1)$. This means that $(\zeta_t)_{t \geqslant 0}$ is an *asymmetric exclusion process* on $\mathbb{Z}$ living on the subset FR, which is invariant for the dynamics described here.

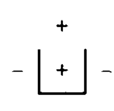
Schematically, the transition rates for the "particles" (the ones) can be depicted as

$$\begin{array}{ccccccccccccc} \cdots & \underset{m}{\overset{M}{\rightleftarrows}} & \bullet & \underset{m}{\overset{M}{\rightleftarrows}} & \bullet & \underset{m}{\overset{M}{\rightleftarrows}} & \bullet & \underset{m}{\overset{M}{\rightleftarrows}} & \bullet & \underset{m}{\overset{M}{\rightleftarrows}} & \bullet & \underset{m}{\overset{M}{\rightleftarrows}} & \cdots \\ \cdots & & -2 & & -1 & & 0 & & 1 & & 2 & & \cdots \end{array}$$

where $m = [1\text{-}\tanh(\alpha)/2$, $M = [1 + \tanh(\alpha)]/2$.

2.2. Partly Frozen Dynamics and Nonmonotone Interfaces

Thinking about an extension of the mapping to one-dimensional processes to configurations with nonmonotone interface heights, one is forced to deal with interfaces containing a segment like



so that flips with $s = 3$ must be allowed which locally cut down the interface length by two units. This makes it difficult to map to one-dimensional model with local interaction. Another open question is the dependence of the resulting equation on $c^\alpha(\cdot, s)$ for $s \geqslant 3$.

If one agrees to modify the dynamics, it is nevertheless possible to study the motion of interfaces composed of a "zigzag stairstep" and another

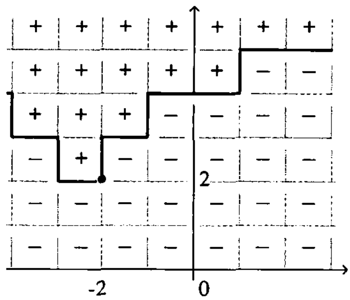


Fig. 2. A V-stairstep.

increasing stairstep like the one in Fig. 2. Here the site at which both parts meet is market by a dot. This configuration satisfies the following definition with $N(\omega) = 2$.

We call $\omega \in \{-1, +1\}^{\mathbb{Z}^2}$ a V-*stairstep* if there is an integer $N(\omega) \geqslant 0$ such that the interface contains the edges

$$((-k-1, k), (-k, k)) \qquad \text{and} \qquad ((-k-1, k+1), (-k-1, k))$$

for all $k \leqslant -N(\omega)$ and runs in the half-plane $[-N(\omega), \infty) \times \mathbb{R}$ only rightward or upward (i.e., it forms an increasing stairstep).

Obviously a V-stairstep determined by $N(\omega)$ and the shape of its increasing stairstep. There is a one-to-one correspondence between the set VS of all V-stairsteps and the set $\mathbb{N}^{\{0\}} \times \{0, 1\}^{\mathbb{Z}_+}$, namely $\omega \in VS$ corresponds to ζ from $\mathbb{N}^{\{0\}} \times \{0, 1\}^{\mathbb{Z}_+}$ if and only if the lattices sites on the increasing stairstep of ω are given by

$$\left(k - \sum_{l=0}^{k} \zeta(l), \sum_{l=0}^{k} \zeta(l)\right)_{k=0,1,2,\ldots} \tag{2.13}$$

The example in Fig. 2 corresponds to

$$\zeta = \underset{0.\ 1.\ 2.\ 3.\ \ldots}{(2, 1, 0, 1, 0, 0, 1, 0, 0, \ldots)}$$

Generally we have $N(\omega) = \zeta(0)$. Horizontal pieces of the interface are again mapped to zeros, vertical ones to ones.

Next we introduce the modified dynamics. On the increasing stairstep part of the interface of $\omega \in VS$ we keep the rates as before, i.e., there are only $(s = 2)$-events with transition rates

$$c^{\alpha}(-1,2) \quad \text{from} \quad \boxed{-} \quad \text{to} \quad \boxed{+}$$

$$c^{\alpha}(+1,2) \quad \text{from} \quad \boxed{+} \quad \text{to} \quad \boxed{-}$$

We do not permit any spin-flip on the decreasing zigzag part of the interface except if there is exactly one positive spin with the lowest height. At such a site we have $s=3$ and we allow the transition

$$\boxed{+} \quad \text{to} \quad \boxed{-} \qquad \text{with rate} \quad c^{\alpha}(+1,3)$$

Every flip of this type lets $N(\omega)$ grow by one unit; all other flips do not affect this number.

One may regard the zigzag part as frozen for the dynamics and the events with $s=3$ as unfreezing at the point where both parts meet.

Since VS is invariant under these dynamics, we can map the spin-flip process $(\omega_t)_{t\geqslant 0}$ to its image $(\zeta_t)_{t\geqslant 0}$ with state space $\mathbb{N}^{\{0\}}\times\{0,1\}^{\mathbb{Z}_+}$. Look at its possible transitions. Particles at site zero can no longer leave [since $N(\omega_t)$ is nondecreasing in time]. A particle at site one may jump to zero leaving a hole behind, increasing $\zeta(0)$ by one unit. Note that this takes place independent of the number of particles at zero, so there is no exclusion restriction. All otherjumps of particles are as in the exclusion process. The mnemonic for the rates reads

$$\begin{array}{ccccccccc}
 & & & \xrightarrow{m} & & \xrightarrow{m} & & \xrightarrow{m} & \\
\bullet & & \bullet & & \bullet & & \bullet & & \cdots \\
 & \xleftarrow[M]{} & & \xleftarrow[M]{} & & \xleftarrow[M]{} & & \xleftarrow[M]{} & \\
0 & & 1 & & 2 & & 3 & & \cdots
\end{array}$$

with again $m=[1-\tanh(\alpha)]/2$, $M=[1+\tanh(\alpha)]/2$. We shall refer to $(\zeta_t)_{t\geqslant 0}$ as the *exclusion process with trap at zero*.

Again, we make α dependent on ε by setting $\alpha=\gamma\varepsilon$ for some $\gamma\neq 0$. Let $(\omega_t^\varepsilon)_{t\geqslant 0}$ and $(\zeta_t^\varepsilon)_{t\geqslant 0}$ denote the corresponding families of processes with state spaces VS and $\mathbb{N}^{\{0\}}\times\{0,1\}^{\mathbb{Z}_+}$, respectively. The rescaled heights are defined as

$$y^\varepsilon(t,a):=\varepsilon\sum_{k=0}^{a/\varepsilon}\zeta^\varepsilon_{t/\varepsilon^2}(k), \qquad a\in\varepsilon\mathbb{N}, \quad t\geqslant 0$$

The lattice points on the increasing part of the rescaled interface are then given by $(x^\varepsilon(t,a), y^\varepsilon(t,a))_{a\in\varepsilon\mathbb{N}}$, where

$$x^\varepsilon(t,a):=a-y^\varepsilon(t,a)=a-\varepsilon\sum_{k=0}^{a/\varepsilon}\zeta^\varepsilon_{t/\varepsilon^2}(k), \qquad a\in\varepsilon\mathbb{N}, \quad t\geqslant 0$$

In the (t, x) coordinates, we expect that the interface height $\varphi(t, x)$ equals $-x$ for x less than some threshold moving left with time due to unfreezing, and that it will be nondecreasing to the right of this threshold.

On that nondecreasing branch, (1.1) simplifies to

$$\varphi_t = \tfrac{1}{2}\varphi_{xx}(1+\varphi_x)^{-2} + \gamma\varphi_x(1+\varphi_x)^{-1}$$

In a fashion similar to that preceding Theorem 2.1, one can check that this is the equation following from

$$\frac{\partial y}{\partial t}(t, a) = \frac{1}{2}\frac{\partial^2 y}{\partial a^2}(t, a) + \gamma \cdot \frac{\partial y}{\partial a}(t, a) - \gamma\left(\frac{\partial y}{\partial a}(t, a)\right)^2, \qquad t>0, \quad a>0 \quad (2.14)$$

after substituting

$$\varphi(t, x) = y(t, a) \text{ for the } x \text{ with } x = a - y(t, a)$$

and eliminating a subsequently.

In analogy to $y^\varepsilon(t, a) = \varepsilon \cdot \zeta^\varepsilon_{t/\varepsilon^2}(0)$, the boundary condition at $a = 0$ should have the form

$$y(t, 0) = q(t), \qquad t>0$$

where $q(t)$ is an asymptotic expression for the height of the bottom of the V-stairstep at macroscopic time t.

We will show that this is true with $q_0 := y_0(0)$ and

$$q(t) := q_0 + \frac{1}{2}\int_0^t \frac{\partial^2 y}{\partial a^2}(s, 0)\, ds, \qquad t>0$$

i.e., the boundary condition will be

$$y(t, 0) = y_0(0) + \frac{1}{2}\int_0^t \frac{\partial^2 y}{\partial a^2}(s, 0)\, ds, \qquad t>0 \qquad (2.15)$$

Here is the analog of Theorem 2.1 for the process with modified dynamics:

Theorem 2.2. Let ω_0^ε, $\varepsilon>0$, be random initial configurations from *VS* with the property that their rescaled heights $y^\varepsilon(0, \cdot)$ converge vaguely in probability to a given piecewise differentiable function y_0: $[0, \infty) \to \mathbb{R}$ in the sense that

$$\varepsilon \sum_{a \in \varepsilon\mathbb{N}} f(a)\, y^\varepsilon(0, a) \xrightarrow{\mathbb{P}} \int_0^\infty f(a)\, y_0(a)\, da \qquad \text{for} \quad \varepsilon \to 0^+ \qquad (2.16)$$

for all f in the space $C_0[0, \infty)$ of continuous functions on $[0, \infty)$ with compact support.

Then $y^\varepsilon(t, a)$ converges vaguely in probability to a solution $y(t, a)$ of the boundary value problem (2.14), (2.15). The convergence is uniform in $[0,T]$ for all finite $T>0$. These statements mean that we have for every $f \in C_0[0, \infty)$ and for arbitrary $T>0$ and $\delta>0$

$$\mathbb{P}\left(\sup_{0\leqslant t\leqslant T}\left|\varepsilon \sum_{a\in\varepsilon\mathbb{Z}} f(a)\, y^\varepsilon(t, a) - \int_{-\infty}^{\infty} f(a)\, y(t, a)\, da\right| > \delta\right)$$
$$\to 0 \qquad \text{as} \quad \varepsilon\to 0^+ \tag{2.17}$$

Furthermore, the initial condition

$$y(t, a) \to y_0(a) \qquad \text{for every} \quad a\geqslant 0 \qquad \text{as} \quad t\to 0^+ \tag{2.18}$$

is satisfied if the derivative of y_0 is Hölder continuous in a neighborhood of $a=0$.

A transformation similar to (2.6) maps the problem (2.14), (2.15), (2.18) to an initial value problem for the heat equation with a mixed boundary condition. This yields an explicit solution $y(t, a)$. We are not going to discuss its uniqueness here.

3. RESULTS ON THE HYDRODYNAMICS OF WEAKLY ASYMMETRIC EXCLUSION PROCESSES

In this short section we collect the result on the hydrodynamics of the one-dimensional exclusion process on $\mathbb{Z}$ and for its version on the non-negative integers with trap at zero.

3.1. Exclusion on the Integers

For any fixed $\gamma\neq 0$, denote by $(\zeta_t^\varepsilon)_{t\geqslant 0}$ the weakly asymmetric exclusion process on $\mathbb{Z}$ (defined for general α in the last section) with $\alpha=\gamma\varepsilon$ and define measures X_t^ε, $t\geqslant 0, \varepsilon>0$, on the real axis by

$$X_t^\varepsilon := \sum_{k=-\infty}^{\infty} \varepsilon\cdot\zeta_{t/\varepsilon^2}^\varepsilon(k)\cdot\delta_{\varepsilon k} \tag{3.1}$$

Here δ_x denotes the Dirac measure concentrated in x. Let u_0 be a piecewise continuous function on $\mathbb{Z}$ with values in $[0,1]$ and denote by $u(t, a)$, $t>0$,

$a \in \mathbb{R}$, the unique bounded solution of *Burgers' equation with viscosity* and initial data u_0

$$\frac{\partial u}{\partial t}(t, a) = \frac{1}{2} \frac{\partial^2 u}{\partial a^2}(t, a) - \gamma \cdot \frac{\partial}{\partial a}(u(t, a)[1 - u(t, a)]), \qquad t > 0, \quad a \in \mathbb{R}$$
$$u(t, a) \to u_0(a) \qquad \text{for} \quad t \to 0^+ \quad \text{at all continuity points } a \text{ of } u_0 \tag{3.2}$$

The measures X_t^ε and $u(t, a)\, da$ both belong to the space $\mathcal{M}_+$ of locally finite measures on $\mathbb{R}$. We equip it with the vague topology, i.e., a sequence of measures $(\mu_n)_{n \in \mathbb{N}}$ from $\mathcal{M}_+$ converges to $\mu \in \mathcal{M}_+$ if and only if

$$\int_{\mathbb{R}} f\, d\mu_n \to \int_{\mathbb{R}} f\, d\mu \qquad \text{for all} \quad f \in C_0$$

Then $\mathcal{M}_+$ is Polish. For details see Kallenberg,[5] Chapter 15, or Bauer,[1] Theorem 31.5.

The following theorem was proved by Gärtner[4]:

Theorem 3.1. Suppose that the distributions of (ζ_0^ε) satisfy

$$X_0^\varepsilon \xrightarrow{\mathbb{P}} u_0(a)\, da \qquad \text{vaguely in } \mathcal{M}_+ \quad \text{as } \varepsilon \to 0$$

Then the convergence

$$X_t^\varepsilon \xrightarrow{\mathbb{P}} u(t, a)\, da \qquad \text{vaguely in } \mathcal{M}_+ \quad \text{as } \varepsilon \to 0$$

holds for every $t > 0$ and uniformly in each time interval $[0,T]$ in the sense that we have

$$\mathbb{P}\left(\sup_{t \in [0, T]} \left| \int \psi\, dX_t^\varepsilon - \int \psi(y)\, u(t, y)\, dt \right| > \delta \right) \to 0 \qquad \text{as} \quad \varepsilon \to 0^+$$

for every $\psi \in C_0$.

Remark. Only the case $\gamma = -1$ is actually treated in ref. 4, but the proof easily carries over to general $\gamma \neq 0$. It involves linearizing Burgers' equation by the Cole–Hopf transformation and a stochastic version of this to treat the exclusion process. Other methods for proving convergence of the particle densities to the solution of Burgers' equation are quite different.[2, 6]

3.2. Exclusion on the Nonnegative Integers with Trap at Zero

Denote by $(\zeta_t^\varepsilon)_{t \geqslant 0}$ the exclusion process with trap at zero as described in Section 2.2, where again $\alpha = \gamma\varepsilon$ for some $\gamma \neq 0$. Define random measures in $\mathscr{M}_+$ by

$$X_t^\varepsilon := \sum_{k=0}^{\infty} \varepsilon \cdot \zeta_{t/\varepsilon^2}^\varepsilon(k) \cdot \delta_{\varepsilon k} \qquad \text{for} \quad t \geqslant 0, \quad \varepsilon > 0$$

and let $u_0 \colon (0, \infty) \to [0, 1]$ be a piecewice continuous function. Consider the unique bounded (classical) solution $u(t, a)$ of the initial boundary-value problem

$$\begin{aligned} \frac{\partial u}{\partial t}(t, a) &= \frac{1}{2}\frac{\partial^2 u}{\partial a^2}(t, a) + \gamma \cdot \frac{\partial}{\partial a}(u(t, a)[1 - u(t, a)]), \qquad a > 0, \quad t > 0 \\ u(t, 0) &= 0, \qquad t > 0 \\ \lim_{t \to 0} u(t, a) &= u_0(a) \qquad \text{at all continuity points } a \text{ of } u_0 \end{aligned} \tag{3.3}$$

Depending on u and some number $\hat{q}_0 \geqslant 0$, define the function

$$\hat{q}(t) := \hat{q}_0 + \frac{1}{2}\int_0^t \frac{\partial u}{\partial a}(s, 0)\, ds, \qquad t \geqslant 0 \tag{3.4}$$

Applying basically the same methods as Gärtner[4] to a stochastic Hopf–Cole transformation adapted to the trap case, the following analog to Theorem 3.1 was proved by the author.[12] Actually, our interest in the exclusion process with trap arose originally from the observation that one can map it to a spin-flip process on V-stairsteps.

Theorem 3.2. If

$$X_0^\varepsilon \xrightarrow{\mathbb{P}} \hat{q}_0 \delta_0 + u_0(a)\, da \qquad \text{as} \quad \varepsilon \to 0^+ \text{ in } \mathscr{M}_+$$

then

$$X_t^\varepsilon \xrightarrow{\mathbb{P}} \hat{q}(t)\, \delta_0 + u(t, a)\, da \qquad \text{as} \quad \varepsilon \to 0^+ \text{ in } \mathscr{M}_+$$

holds for all $t \geqslant 0$, uniformly in $[0, T]$ for every $T > 0$.

Remarks. 1. In the definition of $(\zeta_t^\varepsilon)_{t \geqslant 0}$ we did not specify the rate $c^{\alpha(\varepsilon)}(+1, 3)$ for jumps into the trap. A coupling argument shows that all values greater than or equal to $c^{\alpha(\varepsilon)}(+1, 2)$ lead to the same limit. Here we also allow infinity as a value, which means that flips with $s = 3$ occur instantaneously. In ref. 12 we treated the case $c^{\alpha(\varepsilon)}(+1, 3) = c^{\alpha(\varepsilon)}(+1, 2)$.

2. An approximation of the indicator function of $\{0\}$ yields

$$\varepsilon \cdot \zeta^{\varepsilon}_{t/\varepsilon^2}(0) \xrightarrow{\mathbb{P}} \hat{q}(t) \qquad \text{as} \quad \varepsilon \to 0^+$$

for $t \geqslant 0$, so $\hat{q}(t)$ is the asymptotic mass of particles in the trap at macroscopic time t if it was initially $\hat{q}_0$.

4. DERIVATION OF THE EQUATIONS FOR INTERFACE MOTION

In this section we prove Theorems 2.1 and 2.2. We first explain the idea.

Looking at (2.12) in our scaling of the interface, note that in the case of Theorem 2.1 we have

$$y^{\varepsilon}(t, a) = X^{\varepsilon}_t(a, \infty) = \varepsilon \sum_{k=[a/\varepsilon]+1}^{\infty} \zeta^{\varepsilon}_{t/\varepsilon^2}(k) \tag{4.1}$$

for $t \geqslant 0$ and $a \in \varepsilon\mathbb{Z}$. If the vague convergence of Theorem 3.1 would imply

$$X^{\varepsilon}_t(a, \infty) \xrightarrow{\mathbb{P}} \int_a^{\infty} u(t, z)\, dz \qquad \text{as} \quad \varepsilon \to 0^+ \tag{4.2}$$

we would get convergence of the rescaled heights $y^{\varepsilon}(t, a)$ to

$$\hat{y}(t, a) := \int_a^{\infty} u(t, z)\, dz$$

and Burgers' equation (3.2) would give Eq. (2.5b) for $\hat{y}$. Similar considerations are possible for the trap case.

The tightness condition (2.10) of Theorem 2.1 will allow us to check the assumption of the following lemma, which ensures (4.2) and allows us to carry out the above program. A sketch of proof will be given before we proceed.

Lemma 4.1. In the situation of Theorem 3.1, assume that for some $\varepsilon_0 > 0$ and all $\delta > 0$ we have additionally $\zeta^{\varepsilon}_0 \in FR$ almost surely and

$$\lim_{L \to \infty} \sup_{\varepsilon \leqslant \varepsilon_0} \mathbb{P}(X^{\varepsilon}_0(L, \infty) > \delta) = 0 \tag{4.3}$$

Then the following statements are true:

(i) We have

$$\lim_{L \to \infty} \sup_{\varepsilon \leqslant \varepsilon_0} \mathbb{P}\Big(\sup_{t \in [0, T]} X^{\varepsilon}_t(L, \infty) > \delta\Big) = 0 \quad \text{for every } T > 0 \text{ and } \delta > 0 \tag{4.4}$$

(ii) For every $a \in \mathbb{R}$ and $\delta > 0$,

$$\lim_{t \to 0^+} \overline{\lim_{\varepsilon \to 0^+}} \mathbb{P}(|X_t^\varepsilon(a, \infty) - X_0^\varepsilon(a, \infty)| > \delta) = 0$$

(iii) $u(t, \cdot)$ is integrable on $[0, \infty)$ and we have

$$X_t^\varepsilon(a, \infty) \xrightarrow{\mathbb{P}} \int_a^\infty u(t, z)\, dz \qquad \text{as} \quad \varepsilon \to 0 \quad \text{for all} \quad a \in \mathbb{R}$$

both uniformly in $[0, T]$ for arbitrary $T > 0$.

Sketch of the Proof of Lemma 4.1. That (4.3) implies (i) and (ii) reflects the fact that we are working on a space-time scale on which macroscopic masses move at a finite speed. So it is perhaps not surprising that a proof requires the same tools as the law of large number (Theorem 2.1), namely the nonlinear transformation of the process and the estimation methods of ref. 4. We omit the details.

If $u(t, \cdot)$ were not integrable as stated in (iii), there would be $\delta > 0$ such that for every $L > 0$ there is $t \in [0, T]$ and $M > L$ with

$$\int_L^M u(t, z)\, dz > 2\delta$$

Using Theorem 3.1, we obtain

$$\mathbb{P}(X_t^\varepsilon(L, M) > \delta) \to 1 \qquad \text{for} \quad \varepsilon \to 0^+$$

after approximating the indicator function $\mathbf{1}_{(L, M)}(\cdot)$ by $C_0(\mathbb{R})$-functions. The approximation procedure works because

$$X_t^\varepsilon[L, L+h] \leqslant \varepsilon([h/\varepsilon] + 1) \qquad \text{for all} \quad t \geqslant 0, \quad h \geqslant 0, \quad \text{and} \quad \varepsilon > 0$$

holds almost surely and u is uniformly bounded in $[0, T] \times \mathbb{R}$. This implies for δ chosen as above and arbitrary $L > 0$ and $\varepsilon_0 > 0$

$$\sup_{\varepsilon \leqslant \varepsilon_0} \mathbb{P}(\sup_{t \in [0, T]} X_t^\varepsilon(L, \infty) > \delta) \geqslant 1/2$$

which contradicts (i).

For the remaining $\mathbb{P}$-convergence, fix $\eta > 0$, $\delta > 0$, and $a \in \mathbb{R}$. By (4.4) and the uniform integrability of u there is an $L > 0$ such that

$$\mathbb{P}(\sup_{t \in [0, T]} X_t^\varepsilon(L, \infty) > \delta) > \eta \qquad \text{for all} \quad \varepsilon \leqslant \varepsilon_0$$

and

$$\sup_{t\in[0,T]} \int_L^\infty u(t,z)\,dz > \delta$$

Now approximate the indicator $\mathbf{1}_{(a,L)}(\cdot)$ by compactly supported functions and apply Theorem 3.1. This procedure shows that there is an $\varepsilon_1 > 0$ such that

$$\mathbb{P}\left(\sup_{t\in[0,T]} \left|X_t^\varepsilon(a,L) - \int_a^L u(t,z)\,dz\right| \geqslant \delta\right) < \eta \qquad \text{for all} \quad \varepsilon \leqslant \varepsilon_1$$

For $\varepsilon \leqslant \min(\varepsilon_0, \varepsilon_1)$ we obtain

$$\begin{aligned}
&\mathbb{P}\left(\sup_{t\in[0,T]} \left|X_t^\varepsilon(a,\infty) - \int_a^\infty u(t,z)\,dz\right| \geqslant 3\delta\right) \\
&\quad \leqslant \mathbb{P}\left(\sup_{t\in[0,T]} \left|X_t^\varepsilon(a,L) - \int_a^L u(t,z)\,dz\right| \geqslant \delta\right) \\
&\qquad + \mathbb{P}(\sup_{t\in[0,T]} X_t^\varepsilon(L,\infty) \geqslant \delta) + \mathbb{P}\left(\sup_{t\in[0,T]} \left|\int_L^\infty u(t,z)\,dz\right| \geqslant \delta\right)
\end{aligned}$$

which is less than or equal to 2η, since the third summand is zero.

To complete the proof of (iii), send η and δ to zero. ∎

Proof of Theorem 2.1. Fix y_0 and $(\omega_0^\varepsilon)_{\varepsilon>0}$ according to the assumptions of the theorem and set

$$u_0(a) := -\frac{\partial}{\partial a} y_0(a)$$

This is defined for almost all $a \in \mathbb{R}$ and u_0 is piecewise continuous. Remark 2 to Theorem 2.1 shows that its values are in $[0, 1]$.

Recall that image $(\zeta_t^\varepsilon)_{t\geqslant 0}$ of the spin-flip process $(\omega_t^\varepsilon)_{t\geqslant 0}$ under the bijection from PS to FR described in Section 2 is just the exclusion process started in the image ζ_0^ε of ω_0^ε. Let $(X_t^\varepsilon)_{t\geqslant 0}$ be the measure on $\mathbb{R}$ defined in (3.1) and y^ε the rescaled height process of the interface of $(\omega_t^\varepsilon)_{t\geqslant 0}$. Then we have

$$\zeta_0^\varepsilon(a/\varepsilon) = [\,y^\varepsilon(0, a-\varepsilon) - y^\varepsilon(0,a)]/\varepsilon, \qquad a \in \varepsilon\mathbb{Z}$$

Our first aim is to show the assumption of Theorem 3.1, i.e.,

$$\varepsilon \sum_{k \in \mathbb{Z}} g(\varepsilon k)\, \zeta_0^\varepsilon(k) \xrightarrow{\mathbb{P}} \int_{-\infty}^{\infty} g(a)\, u_0(a)\, da \quad \text{as } \varepsilon \to 0 \text{ for every } g \in C_0 \tag{4.5}$$

To this end, approximate the Dirac distribution concentrated at $a \in \mathbb{R}$ by functions $f \in C_0$ and use (2.9) and the properties of y^ε mentioned in Remark 2 to Theorem 2.1. This results in

$$y^\varepsilon(0, \varepsilon[a/\varepsilon]) \xrightarrow{\mathbb{P}} y_0(a) \qquad \text{as} \quad \varepsilon \to 0^+ \quad \text{for all} \quad a \in \mathbb{R}$$

which can be rewritten as

$$X_0^\varepsilon[a, \infty) \xrightarrow{\mathbb{P}} \int_a^\infty u_0(z)\, dz \qquad \text{as} \quad \varepsilon \to 0^+ \quad \text{for all} \quad a \in \mathbb{R}$$

Next fix arbitrary $g \in C_0$ and $h > 0$ and define

$$g_h(a) := g(h[a/h]) = \sum_{k \in \mathbb{Z}} [g(hk) - g(h(k+1))] \cdot \mathbf{1}_{[hk, \infty)}(a), \qquad a \in \mathbb{R}$$

Clearly it follows that

$$\int g_h(a)\, X_0^\varepsilon(da) \xrightarrow{\mathbb{P}} \int g_h(a)\, u_0(a)\, da, \qquad \varepsilon \to 0^+$$

Sending $h \to 0^+$, this generalizes to (4.5) by standard arguments involving the estimates

$$0 \leqslant X_0^\varepsilon[a, b) \leqslant b - a + \varepsilon \qquad \text{and} \quad 0 \leqslant \int_a^b u_0(z)\, dz \leqslant b - a \qquad \text{for} \quad a < b$$

This shows that we are in the situation of Theorem 3.1. Furthermore, we can apply Lemma 4.1 because (4.3) is just a combination of (2.10) and (4.1).

The interface on the lattice $(\varepsilon\mathbb{Z})^2$ at macroscopic time $t \geqslant 0$ connects the points of the doubly infinite sequence indexed by $a \in \varepsilon\mathbb{Z}$ with members

$$\begin{aligned}(x^\varepsilon(t, a), y^\varepsilon(t, a)) &= \left(a + \varepsilon \sum_{k = a/\varepsilon + 1}^{\infty} \zeta_{t/\varepsilon^2}^\varepsilon(k),\ \varepsilon \sum_{k = a/\varepsilon + 1}^{\infty} \zeta_{t/\varepsilon^2}^\varepsilon(k)\right) \\ &= (a + X_t^\varepsilon(a, \infty), X_t^\varepsilon(a, \infty))\end{aligned}$$

As $\varepsilon \to 0^+$, part (iii) of Lemma 4.1 implies

$$\sup_{t \in [0, T]} \left| y^\varepsilon \left(t, \varepsilon \left[\frac{a}{\varepsilon} \right] \right) - \int_a^\infty u(t, z)\, dz \right| \xrightarrow{\mathbb{P}} 0 \tag{4.6}$$

for all fixed $T > 0$ and $a \in \mathbb{R}$, where u is the solution to Burgers' equation (3.2). Next we prove that

$$\hat{y}(t, a) := \int_a^\infty u(t, z) \qquad \text{for} \quad t > 0, \quad a \in \mathbb{R} \tag{4.7}$$

solves the Cauchy problem (2.5).

By construction and Remark 2 to Theorem 2.1 we have

$$y_0(a) = \int_a^\infty u_0(z)\, dz \to 0 \qquad \text{as} \quad a \to \infty$$

and

$$0 \leqslant u_0(a) \leqslant 1 \qquad \text{almost everywhere on } \mathbb{R}$$

These relations imply for every $T > 0$

$$\sup_{0 \leqslant t \leqslant T} u(t, a) \to 0 \qquad \text{for} \quad a \to \infty \tag{4.8}$$

and

$$\sup_{t_0 \leqslant t \leqslant T} \frac{\partial u}{\partial a}(t, a) \to 0 \qquad \text{as} \quad a \to \infty \quad \text{for every} \quad t_0 \in (0, T] \tag{4.9}$$

Remember that the Cole–Hopf transformation[13] maps Burgers' equation to the heat equation. This provides an explicit solution of (3.2), from which these properties can be extracted. As a side remark, observe that the Cole–Hopf transformation appears implicitly in (2.6): Just replace $y(t, a)$ by (4.7).

Hence the integrals

$$\int_a^\infty \frac{\partial^2 u}{\partial a^2}(t, z)\, dz \qquad \text{and} \qquad \int_a^\infty \frac{\partial u}{\partial a}(t, z)\, dz$$

converge uniformly in $[t_0, T]$. By Burgers' equation, the same is true for

$$\int_a^\infty \frac{\partial u}{\partial t}(t, z)\, dz$$

This ensures

$$\frac{\partial \hat{y}}{\partial t}(t, a) = \int_a^\infty \frac{\partial u}{\partial t}(t, z)\, dz$$

for all $a \in \mathbb{R}$ and $t > 0$ (choose t_0 and T appropriately). Exploiting the definition of $\hat{y}$, Burgers' equation, (4.8), and (4.9) once more, we arrive at

$$\begin{aligned}
\frac{\partial \hat{y}}{\partial t}(t, a) &= \int_a^\infty \left\{\frac{1}{2}\frac{\partial^2 u}{\partial a^2}(t, z) - \gamma \cdot \frac{\partial}{\partial a}(u(t, z)[1 - u(t, z)])\right\} dz \\
&= -\frac{1}{2}\frac{\partial u}{\partial a}(t, a) + \gamma \cdot u(t, a)[1 - u(t, a)] \\
&= \frac{1}{2}\frac{\partial^2 \hat{y}}{\partial a^2}(t, a) - \gamma \cdot \frac{\partial \hat{y}}{\partial a}(t, a) - \gamma \left(\frac{\partial \hat{y}}{\partial a}(t, a)\right)^2
\end{aligned}$$

for arbitrary $t > 0$ and $a \in \mathbb{R}$, i.e., $\hat{y}$ satisfies Eq. (2.5a). The initial condition (2.5b) follows from the triangle inequality

$$\begin{aligned}
&\mathbf{1}\left(\int_a^\infty |u(t, z) - u_0(z)|\, dz > 3\delta\right) \\
&\quad\leqslant \overline{\lim_{\varepsilon \to 0}}\ \mathbb{P}\left(\left|X_t^\varepsilon(a, \infty) - \int_a^\infty u(t, z)\, dz\right| > \delta\right) \\
&\quad\quad + \overline{\lim_{\varepsilon \to 0}}\ \mathbb{P}\left(\left|X_0^\varepsilon(a, \infty) - \int_a^\infty u_0(z)\, dz\right| > \delta\right) \\
&\quad\quad + \overline{\lim_{\varepsilon \to 0}}\ \mathbb{P}(|X_t^\varepsilon(a, \infty) - X_0^\varepsilon(a, \infty)| > \delta), \qquad \delta > 0
\end{aligned}$$

together with the observation that the first two terms on the right vanish by part (iii) of Lemma 4.1, while the third tends to zero as $t \to 0^+$ by part (ii). Sending first t and then δ to zero, we get (2.5b):

$$\hat{y}(t, a) = \int_a^\infty u(t, z)\, dz \to y_0(a) = \int_a^\infty u_0(z)\, dz \qquad \text{as} \quad t \to 0^+ \quad \text{for all} \quad a \in \mathbb{R}$$

So $\hat{y}$ is indeed a solution of problem (2.5).

In order to show that $\hat{y}$ coincides with the function y defined in (2.8), we need to establish

$$\sup_{a \in \mathbb{R}} |y_0(a)|/(1 + |a|) > +\infty$$

and

$$\sup_{0 \leqslant t \leqslant T} \sup_{a \in \mathbb{R}} |\hat{y}(t, a)|/(1+|a|) < +\infty \qquad \text{for every} \quad T > 0$$

in view of the uniqueness result preceding Theorem 2.1.

By construction we have

$$\begin{aligned} &|y^\varepsilon(t, a_1) - y^\varepsilon(t, a_2)| \\ &\qquad \leqslant |a_1 - a_2| \qquad \text{for any} \quad a_1, a_2 \in \varepsilon\mathbb{Z}, \quad t \geqslant 0, \quad \text{and} \quad \varepsilon > 0 \end{aligned}$$

and hence for all $a_1, a_2 \in \mathbb{R}$

$$|y_0(a_1) - y_0(a_2)| \leqslant |a_1 - a_2|$$

and

$$|\hat{y}(t, a_1) - \hat{y}(t, a_2)| \leqslant |a_1 - a_2|, \qquad t \geqslant 0 \tag{4.10}$$

Therefore it suffices to show

$$\sup_{0 \leqslant t \leqslant T} |\hat{y}(t, 0)| < +\infty \qquad \text{for any} \quad T > 0$$

This can be obtained using, for example, the transformation (2.6) onto the heat equation.

So we established $y = \hat{y}$. Equations (4.6) and (4.7) now yield

$$\sup_{t \in [0, T]} \left| y^\varepsilon\left(t, \varepsilon\left[\frac{a}{\varepsilon}\right]\right) - y(t, a)\right| \xrightarrow{\mathbb{P}} 0 \qquad \text{for every} \quad a \in \mathbb{R} \quad \text{and} \quad T > 0$$

Applying (4.10) again, we obtain

$$\sup_{0 \leqslant t \leqslant T} \left| \varepsilon \sum_{a \in \varepsilon\mathbb{Z}} f(a)\, y^\varepsilon(t, a) - \int_{-\infty}^{\infty} f(a)\, y(t, a)\, da \right| \xrightarrow{\mathbb{P}} 0 \qquad \text{as} \quad \varepsilon \to 0^+$$

for all $T > 0$ and every compactly supported continuous f. Thus, we have shown the assertion (2.11) of Theorem 2.1. ∎

Theorem 2.2 can be proved in a similar way exploiting the law of large numbers for the exclusion process with trap given above as Theorem 3.2.

Proof of Theorem 2.2. Recall that $(\omega_t^\varepsilon)_{t \geqslant 0}$ now denotes the spin-flip process with state space VS and partly frozen dynamics and $(\zeta_t^\varepsilon)_{t \geqslant 0}$ the corresponding exclusion process with trap on $\mathbb{N}^{\{0\}} \times \{0, 1\}^{\mathbb{Z}_+}$.

As is seen from (2.13), the $(\varepsilon\mathbb{Z})^2$ lattice points on the increasing part of the interface of ω_t^ε are given by the sequence

$$\begin{aligned}(x^\varepsilon(t, a), y^\varepsilon(t, a)) &= (a - X_t^\varepsilon[0, a], X_t^\varepsilon[0, a]) \\ &= \left(a - \varepsilon \sum_{k=0}^{a/\varepsilon} \zeta_{t/\varepsilon^2}^\varepsilon(k), \varepsilon \sum_{k=0}^{a/\varepsilon} \zeta_{t/\varepsilon^2}^\varepsilon(k)\right)\end{aligned}$$

indexed by $a \in \varepsilon\mathbb{N}$.

Approximate the indicator function $\mathbf{1}_{\{a\}}$ by a sequence from $C_0[0, \infty)$ to obtain from (2.16)

$$y^\varepsilon(0, \varepsilon[a/\varepsilon]) \xrightarrow{\mathbb{P}} y_0(a) \qquad \text{as} \quad \varepsilon \to 0^+ \quad \text{for all} \quad a \geqslant 0$$

Setting

$$u_0(a) := \frac{d}{da} y_0(a) \qquad \text{and} \qquad q_0 := y_0(0)$$

(the first is defined for almost all $a \in [0, \infty)$), we find that this goes over to

$$X_0^\varepsilon[0, a] \xrightarrow{\mathbb{P}} q_0 + \int_0^a u_0(z)\, dz \qquad \text{as} \quad \varepsilon \to 0^+$$

for every $a \geqslant 0$. This extends easily to the vague convergence assumed in Theorem 3.2 (with $\hat{q} := q_0$). Applying this theorem and another approximation, we deduce

$$\sup_{0 \leqslant t \leqslant T} \left| X_t^\varepsilon[0, a] - \hat{q}(t) - \int_0^a u(t, z)\, dz \right| \xrightarrow{\mathbb{P}} 0 \qquad \text{as} \quad \varepsilon \to 0^+$$

for every $a \geqslant 0$ and $T > 0$, where u solves (3.3) and $\hat{q}(t)$ is defined by (3.4). Rewrite this as

$$\sup_{0 \leqslant t \leqslant T} |y^\varepsilon(t, \varepsilon[a/\varepsilon]) - y(t, a)| \xrightarrow{\mathbb{P}} 0 \qquad \text{as} \quad \varepsilon \to 0^+ \tag{4.11}$$

with

$$y(t, a) := \hat{q}(t) + \int_0^a u(t, z)\, dz, \qquad t > 0, \quad a \geqslant 0 \tag{4.12}$$

We are now going to show that y satisfies Eq. (2.14), (2.15), and (2.17) if y_0 is defined by

$$y_0(a) := \hat{q}_0 + \int_0^a u_0(z)\, dz, \qquad a \geqslant 0$$

The boundedness and piecewise continuity of u_0 imply that $\partial u/\partial t$ is uniformly bounded in all regions of the form $[t_0, T] \times [0, A]$, $0 < t_0 < T$, $A \geqslant 0$, so that we can differentiate (4.12):

$$\frac{\partial y}{\partial t}(t, a) = \frac{d}{dt}\hat{q}(t) + \int_0^a \frac{\partial u}{\partial t}(t, z)\, dz, \qquad a \geqslant 0, \quad t > 0$$

Replace the time derivatives on the right-hand side using (3.4) and Burgers' equation (3.3) with its boundary condition. Values at $(t, 0)$ drop out and we get

$$\frac{\partial y}{\partial t}(t, a) = \frac{1}{2}\frac{\partial u}{\partial a}(t, a) + \gamma \cdot u(t, a)[1 - u(t, a)], \qquad t > 0, \quad a \geqslant 0$$

In view of (4.12), this coincides with Eq. (2.14).

From (4.11) one can pass to the vague convergence (2.17) as in the proof of Theorem 2.1.

On the other hand , $q_0 = \hat{q}_0$, (3.4), and (4.11) imply

$$\hat{q}(t) = \hat{q}_0 + \frac{1}{2}\int_0^t \frac{\partial u}{\partial a}(s, 0)\, ds = q_0 + \frac{1}{2}\int_0^t \frac{\partial^2 y}{\partial a^2}(s, 0)\, ds = q(t), \qquad t \geqslant 0$$

This proves that the boundary condition (2.15) holds.

The initial condition (2.18) follows directly from

$$\int_0^a u(t, z)\, dz \to \int_0^a u_0(z)\, dz \qquad \text{as} \quad t \to 0^+ \quad \text{for every} \quad a \geqslant 0$$

(this is just bounded convergence) and

$$\int_0^t \frac{\partial u}{\partial a}(s, 0)\, ds \to 0 \qquad \text{as} \quad t \to 0^+ \tag{4.13}$$

Because we do allow noncompatible initial and boundary conditions [$u_0(0)$ must not be zero], we have to refer again to the explicit solution. It can be seen that (4.13) holds at least if u_0 is piecewise continuous on $[0, \infty)$ and Hölder continuous in a neighborhood of zero (with some exponent).

This concludes the proof of Theorem 2.2. ∎

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REFERENCES

1. H. Bauer, *Mass- und Integrationstheorie* (Berlin, 1990).
2. A. De Masi, E. Presutti, and E. Scacciatelli, The weakly asymmetric exclusion process, *Ann. Inst. H. Poincaré Prob. Stat.* **25**:1–38 (1989).
3. A. De Masi and E. Presutti, *Mathematical Methods for Hydrodynamic Limits* (Springer, Berlin, 1991).
4. J. Gärtner, Convergence towards Burgers' equation and propagation of chaos for weakly asymmetric exclusion processes, *Stoch. Proc. Appl.* **27**:233–260 (1988).
5. O. Kallenberg, *Random measures* (Berlin, 1986).
6. C. Kipnis, S. Olla, and S. R. S. Varadhan, Hydrodynamics and large deviations for simple exclusion processes, *Commun Pure Appl. Math.* **42**:115–137 (1898).
7. Th. M. Liggett, *Interacting Particle Systems* (Berlin, 1985)
8. J.-P. Marchand and Ph. A. Martin, Exclusion process and droplet shape, *J. Stat. Phys.* **44**: 491–504 (1986).
9. H. Rost, Nonequilibrium behavior of a many particle process, *Z. Warsch. Verw. Gebiete* **58**:41–53 (1981).
10. H. Spohn, *Large Scale Dynamics of Interacting Particles* (Berlin; 1991).
11. H. Spohn, Interface motion in models with stochastic dynamics, *J. Stat. Phys.* **71**: 1081–1132 (1993).
12. Th. Strobel, Ein Gesetz der grossen Zahlen für den schwach asymmetrischen Ausschlussprozess mit Randbedingung, Dissertation, Humboldt-Universität, Fachbereich Mathematik, Berlin (1994); see also Th. Strobel, The Burgers equation as hydrodynamic limit of the exclusion process with boundary condition, preprint, Sonderforschungsbereich 256, Universität Bonn (1994).
13. G. B. Whitham, *Linear and Nonlinear Waves* (New York, 1974).